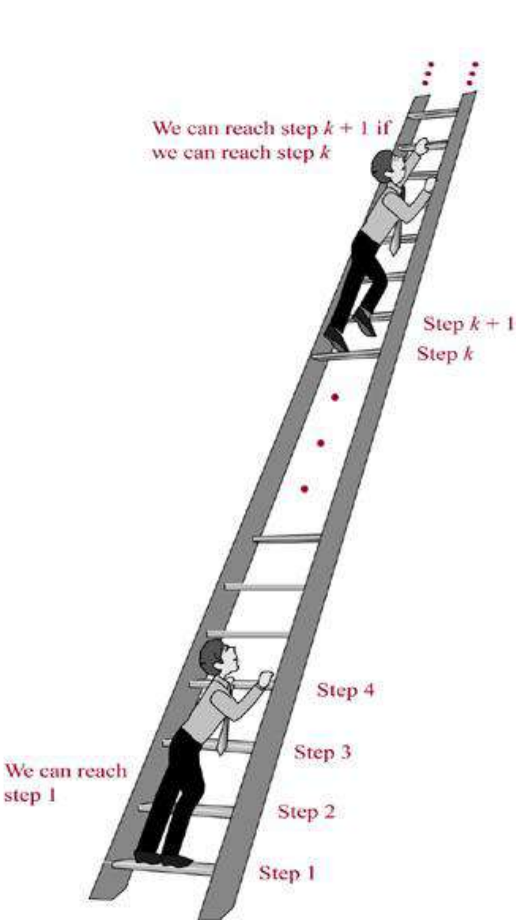
5.1 Mathematical induction



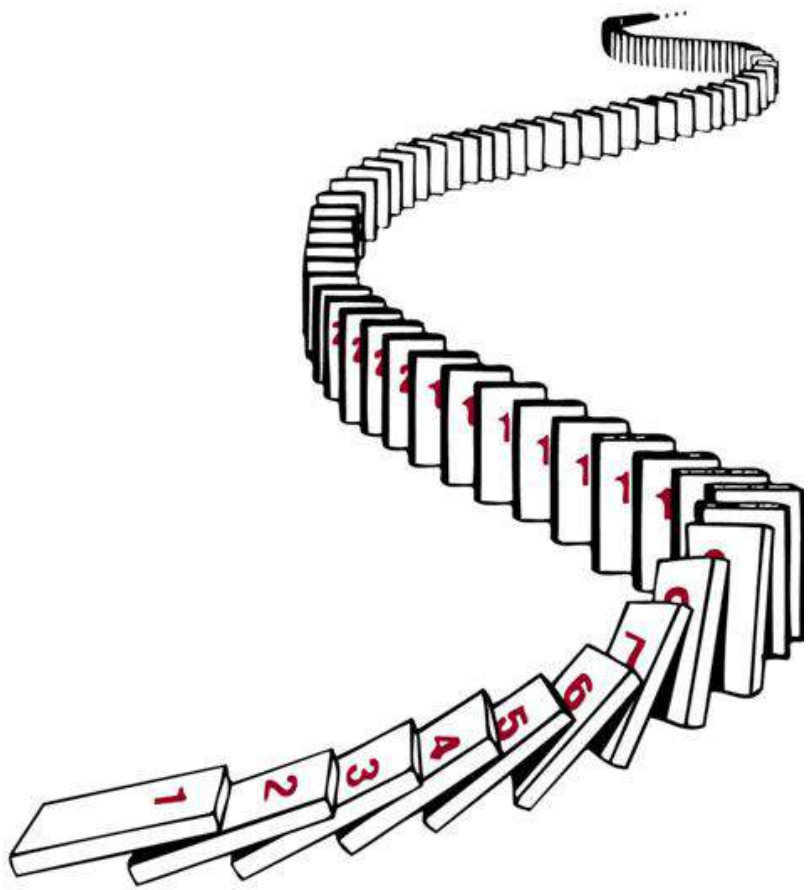
- Want to know whether we can reach *every* step of this ladder
  - We can reach *first* rung of the ladder
  - If we can reach *a particular* run of the ladder, then we can reach *the next* run
- **Mathematical induction**: show that p(n) is true for **every** positive integer n

Mathematical induction

- Two steps
  - **Basis step**: show that p(1) is true
  - **Inductive step**: show that for all positive integers k, if p(k) is true, then p(k+1) is true. That is, we show p(k)→p(k+1) for all positive integers k
- The assumption p(k) is true is called the **inductive hypothesis**
- Proof technique:

$$[p(1) \wedge \forall k(p(k) \rightarrow p(k+1))] \rightarrow \forall n p(n)$$

Analogy



Example

- Show that 1+2+… +n=n(n+1)/2, if n is a positive integer
  - Let p(n) be the proposition that 1+2+… +n=n(n+1)/2
  - Basis step: p(1) is true, because 1=1*(1+1)/2
  - Inductive step: Assume p(k) is true for an arbitrary k. That is, 1+2+…+k=k(k+1)/2

    We must show that 1+2+…+(k+1)=(k+1)(k+2)/2

    From p(k), 1+2+…+k+(k+1)=k(k+1)/2+(k+1)=(k+1)(k+2)/2

    which means p(k+1) is true
  - We have completed the basic and inductive steps, so by mathematical induction we know that p(n) is true for all positive integers n. That is 1+2+…+n=n(n+1)/2

Example

- Conjecture a formula for the sum of the first n positive odd integers. Then prove the conjecture using mathematical induction
- 1=1, 1+3=4, 1+3+5=9, 1+3+5+7=16, 1+3+5+7+9=25
- It is reasonable to conjecture the sum of first n odd integers is n^2, that is, $1+3+5+\ldots+(2n-1)=n^2$
- We need a method to prove whether this conjecture is correct or not

Example

- Let p(n) denote the proposition
- Basic step: $p(1)=1^2=1$
- Inductive steps: Assume that p(k) is true, i.e., $1+3+5+\ldots+(2k-1)=k^2$

  We must show $1+3+5+\ldots+(2k+1)=(k+1)^2$ is true for p(k+1)

  Thus,$1+3+5+\ldots+(2k-1)+(2k+1)=k^2+2k+1=(k+1)^2$ which means p(k+1) is true

  (Note p(k+1) means $1+3+5+\ldots+(2k+1)=(k+1)^2$)
- We have completed both the basis and inductive steps. That is, we have shown p(1) is true and $p(k)\rightarrow p(k+1)$
- Consequently, p(n) is true for all positive integers n

Example

- Use mathematical induction to show that
  $1+2+2^2+\ldots+2^n=2^{n+1}-1$
- Let p(n) be the proposition: $1+2+2^2+\ldots+2^n=2^{n+1}-1$
- Basis step: $p(0)=2^{0+1}-1=1$
- Inductive step: Assume p(k) is true, i.e., $1+2+2^2+\ldots+2^k=2^{k+1}-1$
  It follows
  $(1+2+2^2+\ldots+2^k)+2^{k+1}=(2^{k+1}-1)+2^{k+1}=2*2^{k+1}-1=2^{k+2}-1$ which means $p(k+1)$: $1+2+2^2+\ldots+2^{k+1}=2^{k+2}-1$ is true
- We have completed both the basis and inductive steps. By induction, we show that $1+2+2^2+\ldots+2^n=2^{n+1}-1$

Example

- In the previous step, $p(0)$ is the basis step as the theorem is true $\forall n\, p(n)$ for all non-negative integers
- To use mathematical induction to show that $p(n)$ is true for $n=b, b+1, b+2, \ldots$ where b is an integer other than 1, we show that $p(b)$ is true, and then $p(k) \rightarrow p(k+1)$ for $k=b, b+1, b+2, \ldots$
- Note that b can be negative, zero, or positive

Example

- Use induction to show $\sum_{j=0}^{n} ar^{j} = \frac{ar^{n+1}-a}{r-1} \text{ if } r \neq 1$
- Basis step: p(0) is true as $\frac{ar^{1}-a}{r-1} = \text{a}$
- Inductive step: assume $\sum_{j=0}^{k} ar^{j} = \frac{ar^{k+1}-a}{r-1} \text{ if } r \neq 1$

$$\sum_{j=0}^{k+1} ar^{j} = a + ar + \ldots + ar^{k} + ar^{k+1}$$

$$= \frac{ar^{k+1}-a}{r-1} + ar^{k+1}$$

$$= \frac{ar^{k+1}-a+ar^{k+2}-ar^{k+1}}{r-1}$$

$$= \frac{ar^{k+2}-a}{r-1}$$

- So p(k+1) is true. By induction, p(n) is true for all nonnegative integers

Example

- Use induction to show that $n<2^n$ for $n>0$
- Basis step: p(1) is true as $1<2^1=2$
- Inductive step: Assume p(k) is true, i.e., $k<2^k$

  We need to show $k+1<2^{k+1}$

  $k+1<2^k+1\leq 2^k+2^k=2^{k+1}$

  Thus p(k+1) is true
- We complete both basis and inductive steps, and show that p(n) is true for all positive integers n

Example

- Use induction to show that $2^n<n!$ for $n \geq 4$
- Let p(n) be the proposition, $2^n<n!$ for $n \geq 4$
- Basis step: p(4) is true as $2^4=16<4!=24$
- Inductive step: Assume p(k) is true, i.e., $2^k<k!$ for $k \geq 4$. We need to show that $2^{k+1}<(k+1)!$ for $k \geq 4$

  $2^{k+1} = 2\ 2^k<2\ k!<(k+1)\ k! = (k+1)!$

  This shows p(k+1) is true when p(k) is true
- We have completed basis and inductive steps. By induction, we show that p(n) is true for $n \geq 4$

Example

- Show that n^3-n is divisible by 3 when n is positive
- Basis step: p(1) is true as 1-1=0 is divisible by 3
- Inductive step: Suppose p(k)= k^3-k is true, we must show that $(k+1)^3-(k+1)$ is divisible by 3

  $(k+1)^3-(k+1)=k^3+3k^2+3k+1-(k+1)=(k^3-k)+3(k^2+k)$

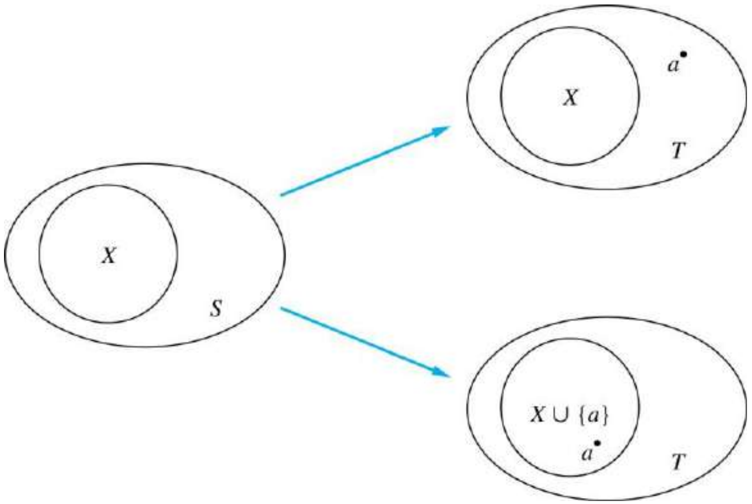
  As both terms are divisible by 3, $(k+1)^3-(k+1)$ is divisible by 3
- We have completed both the basis and inductive steps. By induction, we show that n^3-n is divisible by 3 when n is positive

Example

- Show that if S is a finite set with n elements, then S has 2^n subsets
- Let p(n) be the proposition that a set with n elements has 2^n subsets
- Basis step: p(0) is true as a set with zero elements, the empty set, has exactly 1 subset
- Inductive step: Assume p(k) is true, i.e., S has 2^k subsets if $|S|=k$.

Example

- Let T be a set with k+1 elements. So, $T=S\cup\{a\}$, and $|S|=k$
- For each subset X of S, there are exactly two subsets of T, i.e., X and $X \cup \{a\}$
- Because there are 2^k subsets of S, there are $2 \cdot 2^k = 2^{k+1}$ subsets of T. This finishes the inductive step



Example

- Use mathematical induction to show one of the De Morgan's law: $\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$ where A_1, A_2, …, A_n are subsets of a universal set U, and n≥2
- Basis step: $\overline{A_1 \cap A_2} = \overline{A_1} \cup \overline{A_2}$ (proved Section 2.2, page 131)
- Inductive step: Assume $\overline{\bigcap_{j=1}^{k} A_j} = \bigcup_{j=1}^{k} \overline{A_j}$ is true for k≥2

$$\overline{\bigcap_{j=1}^{k+1} A_j} = \overline{(\bigcap_{j=1}^{k} A_j) \cap A_{k+1}} = \overline{(\bigcap_{j=1}^{k} A_j)} \cup \overline{A_{k+1}}$$

$$= (\bigcup_{j=1}^{k} \overline{A_j}) \cup \overline{A_{k+1}} = \bigcup_{j=1}^{k+1} \overline{A_j}$$

Axioms for the set of positive integers

- See appendix 1
- Axiom 1: The number 1 is a positive integer
- Axiom 2: If n is a positive integer, then n+1, the successor of n, is also a positive integer
- Axiom 3: Every positive integer other than 1 is the successor of a positive integer
- Axiom 4: **Well-ordering property** Every non-empty subset of the set of positive integers has a least element

Why mathematical induction is valid?

- From mathematical induction, we know p(1) is true and the proposition p(k)→p(k+1) is true for all positive integers
- To show that p(n) must be true for all positive integers, assume that there is at least one positive integer such that p(n) is false
- Then the set S of positive integers for which p(n) is false is non-empty
- By well-ordering property, S has a least element, which is demoted by m
- We know that m cannot be 1 as p(1) is true
- Because m is positive and greater than 1, m-1 is a positive integer

Why mathematical induction is valid?

- Because m-1 is less than m, it is not in S
- So p(m-1) must be true
- As the conditional statement p(m-1)→p(m) is also true, it must be the case that p(m) is true
- This contradicts the choice of m
- Thus, p(n) must be true for every positive integer n

Template for inductive proof

Template for Proofs by Mathematical Induction

1. Express the statement that is to be proved in the form "for all $n \geq b$, $P(n)$" for a fixed integer b.
2. Write out the words "Basis Step." Then show that $P(b)$ is true, taking care that the correct value of b is used. This completes the first part of the proof.
3. Write out the words "Inductive Step."
4. State, and clearly identify, the inductive hypothesis, in the form "assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$."
5. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what $P(k+1)$ says.
6. Prove the statement $P(k+1)$ making use the assumption $P(k)$. Be sure that your proof is valid for all integers k with $k \geq b$, taking care that the proof works for small values of k, including $k = b$.
7. Clearly identify the conclusion of the inductive step, such as by saying "this completes the inductive step."
8. After completing the basis step and the inductive step, state the conclusion, namely that by mathematical induction, $P(n)$ is true for all integers n with $n \geq b$.

5.2 Strong induction and well-ordering

- **Strong induction**: To prove p(n) is true for all positive integers n, where p(n) is a propositional function, we complete two steps
- Basis step: we verify that the proposition p(1) is true
- Inductive step: we show that the conditional statement $(p(1) \wedge p(2) \wedge \ldots \wedge p(k)) \rightarrow p(k+1)$ is true for all positive integers k

Strong induction

- Can use all k statements, p(1), p(2), …, p(k) to prove p(k+1) rather than just p(k)
- Mathematical induction and strong induction are equivalent
- Any proof using mathematical induction can also be considered to be a proof by strong induction (induction → strong induction)
- It is more awkward to convert a proof by strong induction to one with mathematical induction (strong induction → induction)

Strong induction

- Also called the **second principle of mathematical induction** or **complete induction**
- The principle of mathematical induction is called **incomplete induction**, a term that is somewhat misleading as there is nothing incomplete
- Analogy:
  - If we can reach the first step
  - For every integer k, if we can reach all the first k steps, then we can reach the k+1 step

Example

- Suppose we can reach the 1st and 2nd rungs of an infinite ladder
- We know that if we can reach a rung, then we can reach two rungs higher
- Can we prove that we can reach every rung using the principle of mathematical induction? or strong induction?

Example – mathematical induction

- Basis step: we verify we can reach the 1^{st} rung
- Attempted inductive step: the inductive hypothesis is that we can reach the k-th rung
- To complete the inductive step, we need to show that we can reach k+1-th rung based on the hypothesis
- However, no obvious way to complete this inductive step (because we do not know from the given information that we can reach the k+1-th rung from the k-th rung)

Example – strong induction

- Basis step: we verify we can reach the 1^{st} rung
- Inductive step: the inductive hypothesis states that we can reach each of the first k rungs
- To complete the inductive step, we need to show that we can reach k+1-th rung
- We know that we can reach 2^{nd} rung.
- We note that we can reach the (k+1)-th rung from (k-1)-th rung we can climb 2 rungs from a rung that we already reach
- This completes the inductive step and finishes the proof by strong induction

Which one to use

- Try to prove with mathematical induction first
- Unless you can clearly see the use of strong induction for proof

5.2 Strong induction and well-ordering

- Use strong induction to show that if n is an integer greater than 1, then n can be written as the product of primes
- Let p(n) be the proposition that n can be written as the product of primes
- <u>Basis step</u>: p(2) is true as 2 can be written as the product of one prime, itself
- <u>Inductive step</u>: Assume p(k) is true with the assumption that p(j) is true for $j \leq k$

Proof with strong induction

- That is, j ($j \le k$) can be written as a product of primes
- To complete the proof, we need to show $p(k+1)$ is true (i.e., $k+1$ can be written as a product of primes)
- There are two cases: when $k+1$ is prime or composite
- If $k+1$ is prime, we immediately see that $p(k+1)$ is true

Proof with strong induction

- If k+1 is composite and can be written as a product of two positive integers a and b, with $2 \le a \le b < k+1$
- By inductive hypothesis, both a and b can be written as product of primes
- Thus, if k+1 is composite, it can be written as the product of primes, namely, the primes in the factorization of a and those in the factorization of b

Proof with induction

- Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps
- First use mathematical induction for proof
- Basis step: Postage of 12 cents can be formed using 3 4-cent stamps
- Inductive step: The inductive hypothesis assumes p(k) is true
- That is, we need to sure p(k+1) is true when $k \geq 12$

Proof induction

- Suppose that at least one 4-cent stamp is used to form postage of k cents
- We can replace this stamp with 5-cent stamp to form postage of k+1 cents
- If no 4-cent stamps are used, we can form postage of k cents using only 5-cent stamps
- As $k \geq 12$, we need at least 3 5-cent stamps to form postage of k cents
- So, we can replace 3 5-cent stamps with 4 4-cent stamps for k+1 cents
- As we have completed basis and inductive steps, we know p(n) is true for $n \geq 12$

Proof with strong induction

- Use strong induction for proof
- In the basis step, we show that p(12), p(13), p(14) and p(15) are true
- In the inductive step, we show that how to get postage of $k+1$ cents for $k \geq 15$ from postage of $k-3$ cents
- Basis step: we can form postage of 12, 13, 14, 15 cents using 3 4-cent stamps, 2 4-cent/1 5-cent stamps, 2 5-cent/1 4-cent stamps, and 3 5-cent stamps. So p(12), p(13), p(14), p(15) are true

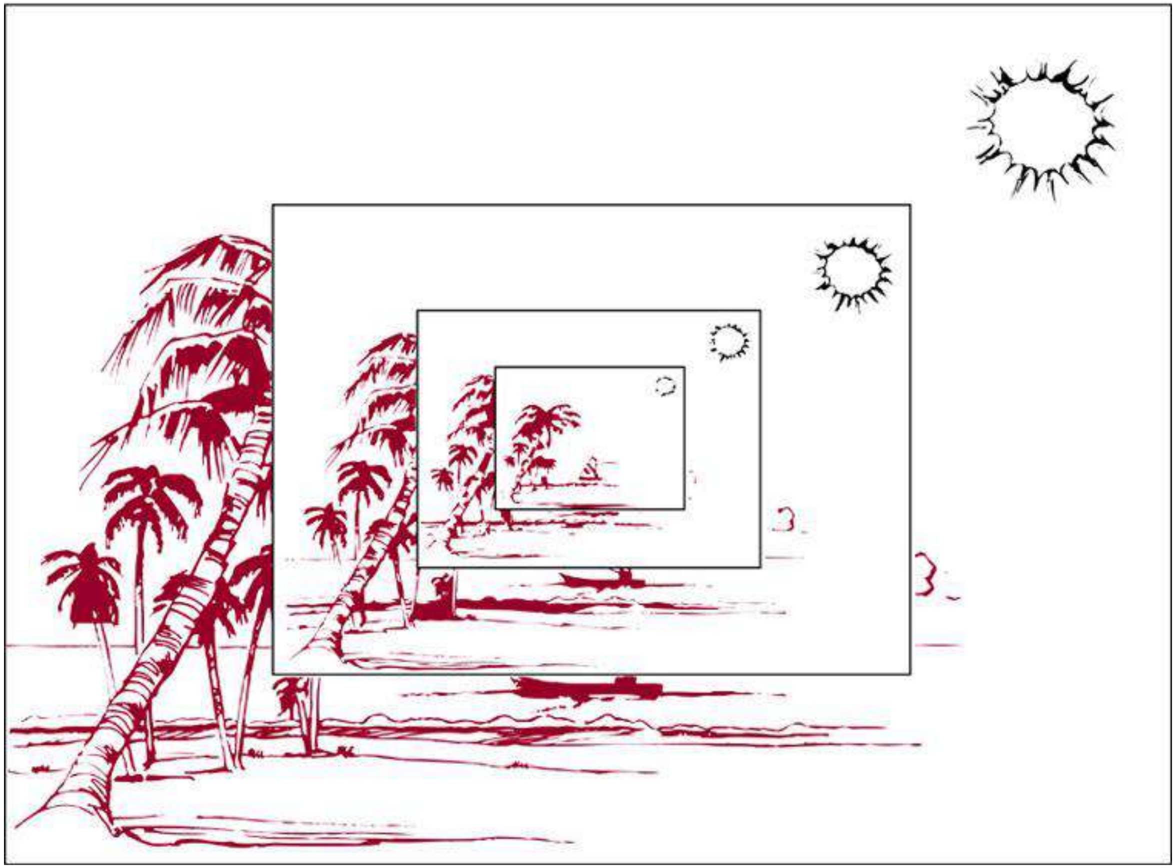
Proof with strong induction

- Inductive step: The inductive hypothesis is the statement p(j) is true for 12≤j ≤k, where k is an integer with k≥15. We need to show p(k+1) is true
- We can assume p(k-3) is true because k-3 ≥12, that is, we can form postage of k-3 cents using just 4-cent and 5-cent stamps
- To form postage of k+1 cents, we need only add another 4-cent stamp to the stamps we used to form postage of k-3 cents. That is, we show p(k+1) is true
- As we have completed basis and inductive steps of a strong induction, we show that p(n) is true for n≥12
- There are other ways to prove this

Proofs using well-ordering property

- Validity of both the principle of mathematical induction and strong induction follows from a fundamental axiom of the set of integers, the well-ordering property
- **Well-order property**: every non-empty set of non-negative integers has a least element
- The well-ordering property can be used directly in proofs
- The **well-ordering property**, the **principle of mathematical induction**, and **strong induction** are all equivalent

5.3 Recursive definitions and structural induction



A recursively defined picture

Recursive definitions

- The sequence of powers of 2 is given by $a_n=2^n$ for $n=0, 1, 2, \ldots$
- Can also be defined by $a_0=1$, and a rule for finding a term of the sequence from the previous one, i.e., $a_{n+1}=2a_n$
- Can use induction to prove results about the sequence
- **Structural induction**: We define a set recursively by specifying some initial elements in a basis step and provide a rule for constructing new elements from those already in the recursive step

Recursively defined functions

- Use two steps to define a function with the set of non-negative integers as its domain
- **Basis step**: specify the value for the function at zero
- **Recursive step**: give a rule for finding its value at an integer from its values at smaller integers
- Such a definition is called a recursive or inductive definition

Example

- Suppose f is defined recursively by
  - $f(0)=3$
  - $f(n+1)=2f(n)+3$

  Find f(1), f(2), f(3), and f(4)
  - $f(1)=2f(0)+3=2*3+3=9$
  - $f(2)=2f(1)+3=2*9+3=21$
  - $f(3)=2f(2)+3=2*21+3=45$
  - $f(4)=2f(3)+3=2*45+3=93$

Example

- Give an inductive definition of the factorial function $f(n)=n!$
- Note that $(n+1)!=(n+1)\cdot n!$
- We can define $f(0)=1$ and $f(n+1)=(n+1)f(n)$
- To determine a value, e.g., $f(5)=5!$, we can use the recursive function

$$f(5)=5\cdot f(4)=5\cdot 4\cdot f(3)=5\cdot 4\cdot 3\cdot f(2)=5\cdot 4\cdot 3\cdot 2\cdot f(1)$$
$$=5\cdot 4\cdot 3\cdot 2\cdot 1\cdot f(0)=5\cdot 4\cdot 3\cdot 2\cdot 1\cdot 1=120$$

Recursive functions

- Recursively defined functions are well defined
- For every positive integer, the value of the function is determined in an unambiguous way
- Given any positive integer, we can use the two parts of the definition to find the value of the function at that integer
- We obtain the same value no matter how we apply two parts of the definition

Example

- Given a recursive definition of a^n, where a is a non-zero real number and n is a non-negative integer
- Note that $a^{n+1}=a \cdot a^n$ and $a^0=1$
- These two equations uniquely define a^n for all non-negative integer n

Example

- Given a recursive definition of $\sum_{k=0}^{n} a_k$
- The first part of the recursive definition

$$\sum_{k=0}^{0} a_k = a_0$$

- The second part is

$$\sum_{k=0}^{n+1} a_k = \left(\sum_{k=0}^{n} a_k\right) + a_{n+1}$$

Example – Fibonacci numbers

- Fibonacci numbers f_0, f_1, f_2, are defined by the equations, $f_0=0$, $f_1=1$, and $f_n=f_{n-1}+f_{n-2}$ for $n=2, 3, 4, \ldots$
- By definition

$f_2=f_1+f_0=1+0=1$

$f_3=f_2+f_1=1+1=2$

$f_4=f_3+f_2=2+1=3$

$f_5=f_4+f_3=3+2=5$

$f_6=f_5+f_4=5+3=8$

Example

- Use strong induction to show when $n \geq 3$, $f_n > \alpha^{n-2}$ where f_n is a Fibonacci number and $\alpha = (1+\sqrt{5})/2$
- Let p(n) be the proposition that $f_n > \alpha^{n-2}$
- Basis step: note that

  $$\alpha < 2 = f_3, \alpha^2 = (3+\sqrt{5})/2 < 3 = f_4$$

  so that p(3) and p(4) are true
- Inductive step: assume p(j) is true, i.e., $f_j > \alpha^{j-2}$ with $3 \leq j \leq k$ where $k \geq 4$. We need to show that p(k+1) is true, i.e., $f_k > \alpha^{k-2}$

Example

- First note that α is a solution to $x^2-x-1=0$, so $\alpha^2=\alpha+1$, thus

$$\alpha^{k-1} = \alpha^2\alpha^{k-3} = (\alpha+1)\alpha^{k-3} = \alpha^{k-2} + \alpha^{k-3}$$

- By inductive hypothesis, if $k\geq 4$, it follows $f_{k-1}>\alpha^{k-3}$, $f_k>\alpha^{k-2}$
- So, $$f_{k+1} = f_k + f_{k-1} > \alpha^{k-2} + \alpha^{k-3} = \alpha^{k-1}$$

  It follows that p(k+1) is true. This completes the proof

Recursively defined sets and structures

- Consider the subset S of the set of integers defined by
  - Basis step: $3 \in S$
  - Recursive step: if $x \in S$ and $y \in S$, then $x+y \in S$
- The new elements formed by this are $3+3=6$, $3+6=9$, $6+6=12$, ...
- We will show that S is the set of all positive multiples of 3 (using structural induction)

String

- The set $\sum^*$ of strings over the alphabet $\sum$ can be defined recursively by
  - Basis step: $\lambda \in \sum^*$ (where λ is the empty string containing no symbols)
  - Recursive step: if $w \in \sum^*$ and $x \in \sum$ then $wx \in \sum^*$
- The basis step defines that the empty string belongs to string
- The recursive step states new strings are produced by adding a symbol from $\sum$ to the end of stings in $\sum^*$
- At each application of the recursive step, strings containing one additional symbol are generated

Example

- If $\sum=\{0, 1\}$, the strings found to be in $\sum^*$, the set of all bit strings, are
- λ, specified to be in $\sum^*$ in the basis step
- 0 and 1 found in the 1^{st} recursive step
- 00, 01, 10, and 11 are found in the 2^{nd} recursive step, and so on

Concatenation

- Two strings can be combined via the operation of concatenation
- Let $\sum$ be a set of symbols and $\sum^*$ be the set of strings formed from symbols in $\sum$
- We can define the concatenation for two strings by recursive steps
  - Basis step: if $w \in \sum^*$, then $w \cdot \lambda = w$, where λ is the empty string
  - Recursive step: If $w_1 \in \sum^*$, $w_2 \in \sum^*$ and $x \in \sum$, then $w_1 \cdot (w_2 x) = (w_1 \cdot w_2)x$
  - Oftentimes $w_1 \cdot w_2$ is rewritten as $w_1 w_2$
  - e.g., w_1=abra, and w_2=cadabra, $w_1 w_2$=abracadabra

Length of a string

- Give a recursive definition of l(w), the length of a string w
- The length of a string is defined by
  - $l(\lambda)=0$
  - $l(wx)=l(w)+1$ if $w \in \sum^*$ and $x \in \sum$

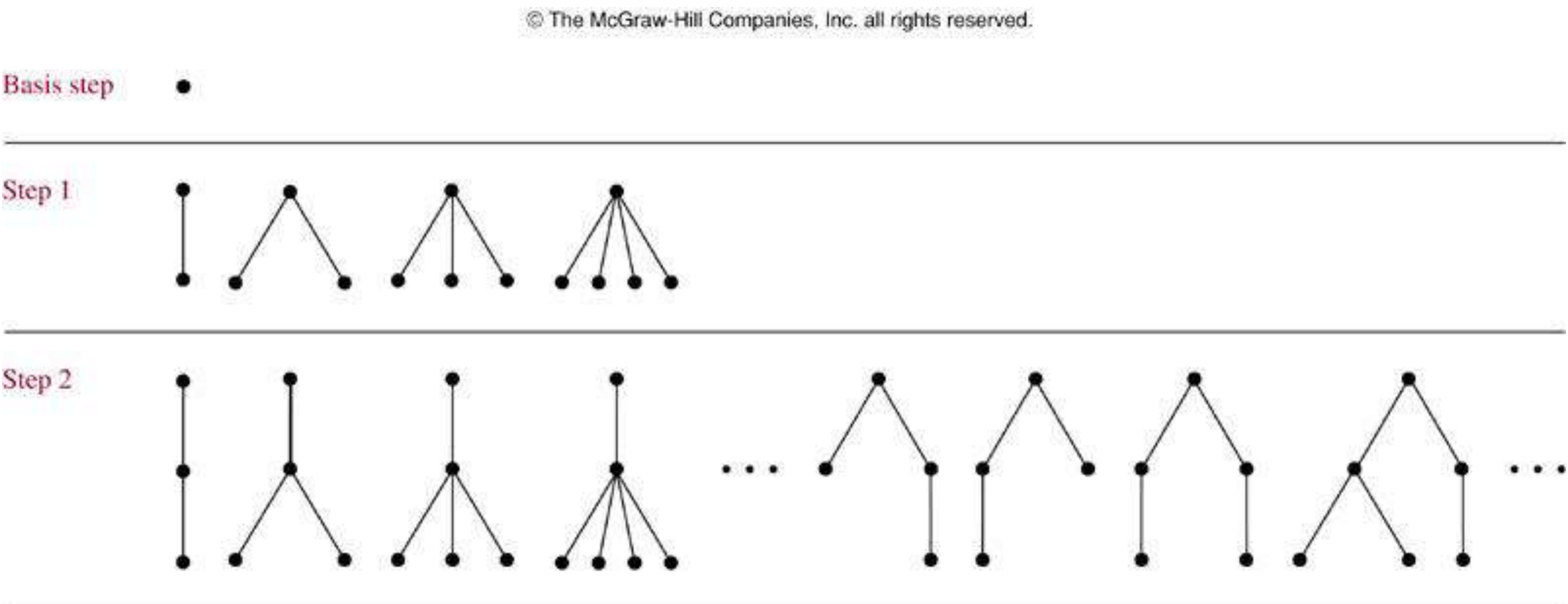
Well-formed formulae

- We can define the set of **well-formed formulae** for compound statement forms involving T, F, proposition variables and operators from the set $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$
- Basis step: T, F, and s, where s is a propositional variable are well-formed formulae
- Recursive step: If E and F are well-formed formulae, then $\neg E$, $E \wedge F$, $E \vee F$, $E \rightarrow F$, $E \leftrightarrow F$ are well-formed formulae
- From an initial application of the recursive step, we know that $(p \vee q)$, $(p \rightarrow F)$, $(F \rightarrow q)$ and $(q \wedge F)$ are well-formed formulae
- A second application of the recursive step shows that $((p \vee q) \rightarrow (q \wedge F))$, $(q \vee (p \vee q))$, and $((p \rightarrow F) \rightarrow T)$ are well-formed formulae

Rooted trees

- The set of rooted trees, where a rooted tree consists of a set of vertices containing a distinguished vertex called the root, and edges connecting these vertices, can be defined recursively by
  - Basis step: a single vertex r is a rooted tree
  - Recursive step: suppose that T_1, T_2, …, T_n are disjoint rooted trees with roots r_1, r_2, …, r_n, respectively.
  - Then the graph formed by starting with a root r, which is not in any of the rooted trees T_1, T_2, …, T_n, and adding an edge from r to each of the vertices r_1, r_2, …, r_n, is also a rooted tree

Rooted trees



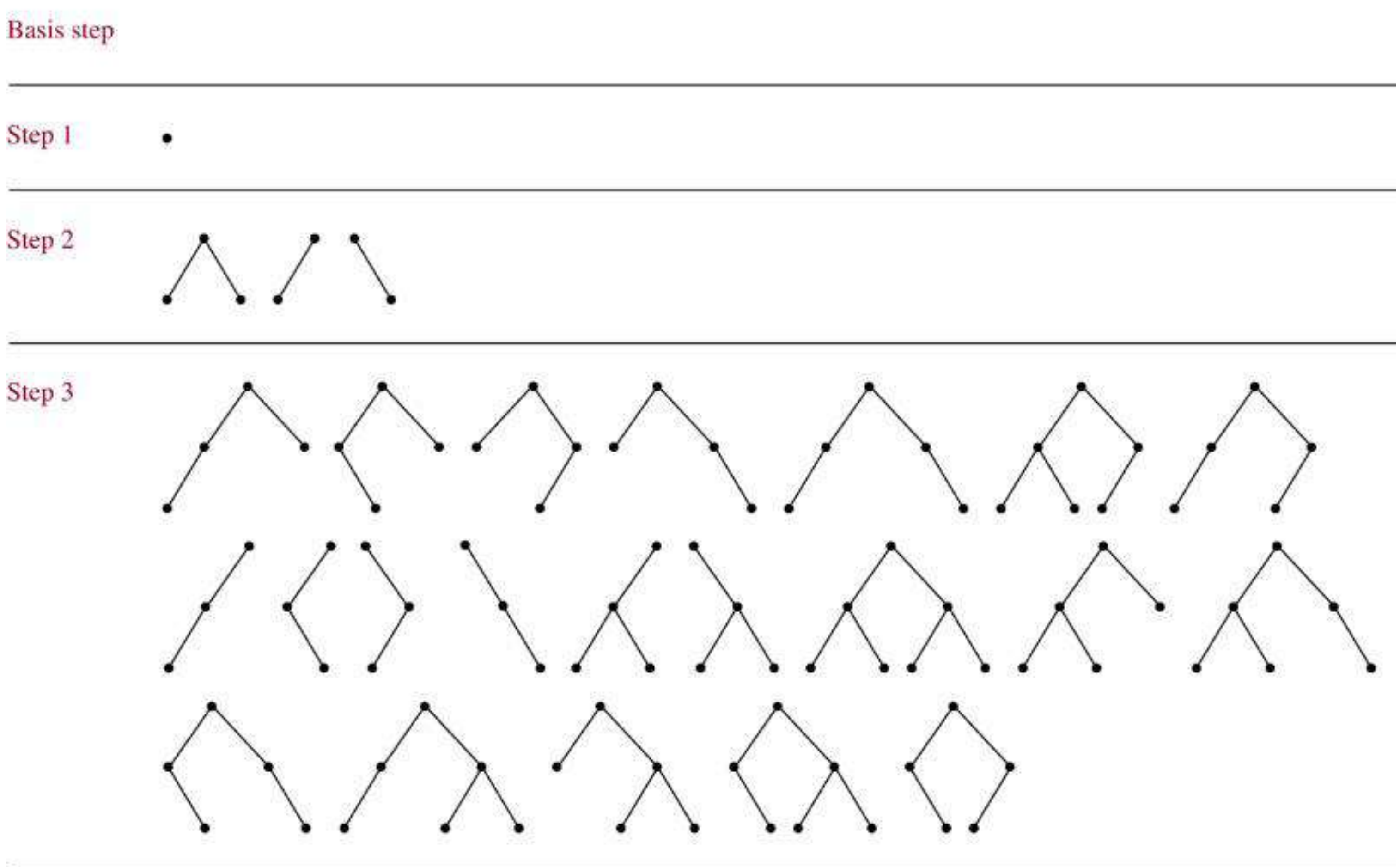
Binary trees

- At each vertex, there are at most two branches (one left subtree and one right subtree)
- Extended binary trees: the left subtree or the right subtree can be empty
- Full binary trees: must have left and right subtrees

Extended binary trees

- The set of extended binary trees can be defined by
  - Basis step: the empty set is an extended binary tree
  - Recursive step: If T_1 and T_2 are disjoint extended binary trees, there is an extended binary tree, denoted by $T_1 \cdot T_2$, consisting of a root r together with edges connecting the root to each of the roots of the left subtree T_1 and right subtree T_2, when these trees are non-empty

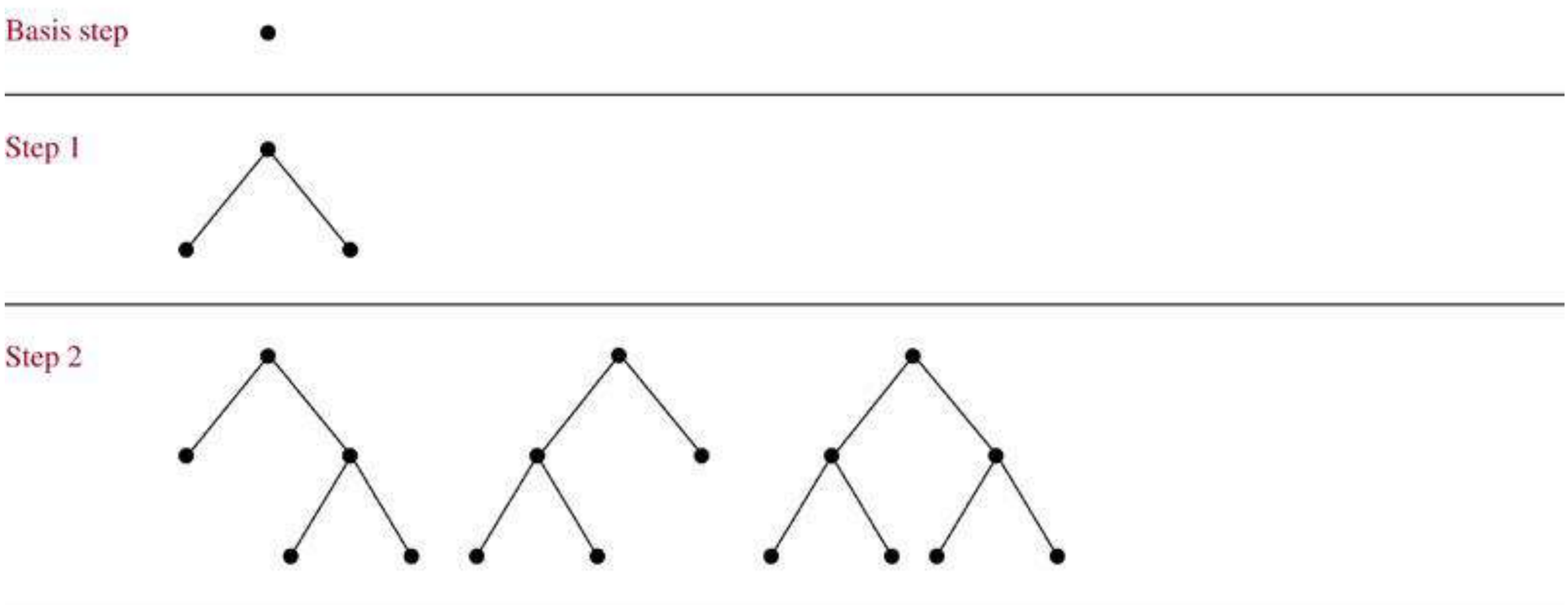
Extended binary trees



Full binary trees

- The set of full binary trees can be defined recursively
  - Basis step: There is a full binary tree consisting only of a single vertex r
  - Recursive step: If T_1 and T_2 are disjoint full binary trees, there is a full binary tree, denoted by $T_1 \cdot T_2$, consisting of a root r together with edges connecting the root to each of the roots of the left subtree T_1 and right subtree T_2

Full binary tree



Structural induction

- Show that the set S defined by
  - $3 \in S$ and
  - if $x \in S$ and $y \in S$, then $x + y \in S$,

  is the set of multiples of 3
- Let A be the set of all positive integers divisible by 3
- To prove A=S, we must show that $A \subseteq S$, and $S \subseteq A$
- To show $A \subseteq S$, we must show that every positive integer divisible by 3 is in S
- Use mathematical induction to prove it

Structural induction

- Let $p(n)$ be the statement that $3n$ belongs to S
- Basis step: it holds as the first part of recursive definition of S, $3 \cdot 1 = 3 \in S$
- Inductive step: assume that $p(k)$ is true, i.e., $3k$ is in S. As $3k \in S$ and $3 \in S$, it follows from the 2nd part of the recursive definition of S that $3k+3=3(k+1) \in S$. So $p(k+1)$ is true

Structural induction

- To show that $S \subseteq A$, we use recursive definition of S
- The basis step of the definition specifies that 3 is in S
- As $3=3\cdot 1$, all elements specified to be in S in this step are divisible by 3, and there in A
- To finish the proof, we need to show that all integers in S generated using the 2nd part of the recursive definition are in A
- This consists of showing that x+y is in A whenever x and y are elements of S also assumed to be in A
- If x and y are both in A, it follows that $3|x$, $3|y$, and thus $3|x+y$, thereby completing the proof

Trees and structural induction

- To prove properties of trees with structural induction
  - Basis step: show that the result is true for the tree consisting of a single vertex
  - Recursive step: show that if the result is true for the trees T_1 and T_2, then it is true for $T_1 \cdot T_2$, consisting of a root r, which has T_1 as its left subtree and T_2 as its right subtree

Height of binary tree

- We define the height $h(T)$ of a full binary tree T recursively
  - Basis step: the height of the full binary tree T consisting of only a root r is $h(T)=0$
  - Recursive step: If T_1 and T_2 are full binary trees, then the full binary tree $T= T_1\cdot T_2$ has height $h(T)=1+\max(h(T_1), h(T_2))$

Number of vertices in a binary tree

- If we let $n(T)$ denote the number of vertices in a full binary tree, we observe that $n(T)$ satisfies the following recursive formula:
  - Basis step: the number of vertices $n(T)$ of the full binary tree consisting of only a root r is $n(T)=1$
  - Recursive step: If T_1 and T_2 are full binary trees, then the number of vertices of the full binary tree $T= T_1 \cdot T_2$ is $n(T)=1+n(T_1)+n(T_2)$

Theorem

- If T is a full binary tree T, then $n(T) \leq 2^{h(T)+1}-1$
- Use structural induction to prove this
- Basis step: for the full binary tree consisting of just the root r the result is true as $n(T)=1$ and $h(T)=0$, so $n(T)=1 \leq 2^{0+1}-1=1$
- Inductive step: For the inductive hypothesis we assume that $n(T_1) \leq 2^{h(T_1)+1}-1,\ n(T_2) \leq 2^{h(T_2)+1}-1$

  where T_1 and T_2 are full binary trees

Theorem

- By the recursive formulae for n(T) and h(T), we have $n(T)=1+n(T_1)+n(T_2)$ and $h(T)=1+\max(h(T_1), h(T_2))$
- Thus,

$$\begin{aligned} n(T) &= 1 + n(T_1) + n(T_2) \\ &\le 1 + (2^{h(T_1)+1} - 1) + (2^{h(T_2)+1} - 1) \\ &\le 2 \cdot \max(2^{h(T_1)+1}, 2^{h(T_2)+1}) - 1 \\ &= 2 \cdot 2^{\max(h(T_1), h(T_2))+1} - 1 \\ &= 2 \cdot 2^{h(T)} - 1 \\ &= 2^{h(T)+1} - 1 \end{aligned}$$

- This completes the inductive step